

Duality Products of Multivectors and Multiforms, and Extensors

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Abstract

In this paper we study in details the properties of the duality product of multivectors and multiforms (used in the definition of the hyperbolic Clifford algebra of *multivefors*) and introduce the theory of the k *multivector* and l *multiform variables multivector* (or *multiform*) *extensors* over V (defining the spaces $ext_k^l(V)$ and $ext_k^{*l}(V)$) studying their properties with considerable detail.

1 Introduction

In this article we first briefly recall the exterior algebras of multivectors (elements of $\bigwedge V$) and multiforms (elements of $\bigwedge V^*$) associated with a real vector space V of finite dimension. Next we explore the structure of the duality product $<, >$ of multivectors by multiforms used in the definition of the hyperbolic Clifford algebra $\mathcal{Cl}(V \oplus V^*, <, >)$ of *multivectors* [5]. We detail some important properties of the left and right contracted products among the elements of $\bigwedge V$ and $\bigwedge V^*$. Next, we give a theory of the *k multivector and l multiform variables multivector (or multiform) extensors* over V (defining the spaces $ext_k^l(V)$ and $ext_k^{*l}(V)$) introducing the concept of exterior product of extensors, and of several operators acting on these objects as, e.g., the adjoint operator, the extension operator, and the generalized operator procedure. We study the properties of these operators with considerable detail. Extensors *fields* (with representatives that are appropriate mappings from an open set $U \subset M$ to $ext_k^l(V)$ or $ext_k^{*l}(V)$) and the operators acting on them will be shown in forthcoming papers to provide a valuable and simplifying tool (and an improvement over a previous presentation [1, 2, 3, 4] which uses only the multivector and extensor calculus) in the study of some of the fundamental ingredients of the differential geometry¹ of an arbitrary manifold M equipped with an arbitrary connection.

2 Multivectors and Multiforms

Let V be a vector space over \mathbb{R} with finite dimension, i.e., $\dim V = n$ with $n \in \mathbb{N}$, and let V^* be its dual vector space. Recall that

$$\dim V = \dim V^* = n. \quad (1)$$

Let us consider an integer number k with $0 \leq k \leq n$. The real vector spaces of k -vectors over V , i.e., the set of skew-symmetric k -contravariant tensors over V , and the real vector spaces of k -forms over V , i.e., the set

¹A presentation of the differential geometry of an arbitrary manifold (admitting a metric field) using the Clifford bundle formalism may be found in [6].

of skew-symmetric k -covariant tensors over V , will be as usually denoted by $\bigwedge^k V$ and $\bigwedge^k V^*$, respectively.

We identify, as usual 0-vectors to real numbers, i.e., $\bigwedge^0 V = \mathbb{R}$, and 1-vectors to objects living in V , i.e., $\bigwedge^1 V \simeq V$. Also, we identify 0-forms with real numbers, i.e., $\bigwedge^0 V = \mathbb{R}$, and 1-forms with objects living in V^* , i.e., $\bigwedge^1 V^* = V^*$. Recall that

$$\dim \bigwedge^k V = \dim \bigwedge^k V^* = \binom{n}{k}. \quad (2)$$

The 0-vectors, 2-vectors, ..., $(n-1)$ -vectors and n -vectors are called scalars, bivectors, ..., pseudovectors and pseudoscalars, respectively. The 0-forms, 2-forms, ..., $(n-1)$ -forms and n -forms are called scalars, bi-forms, ..., pseudoforms and pseudoscalars.

Given a vector space V over the real field \mathbb{R} , we define $\bigwedge V$ as the exterior direct sum

$$\bigwedge V = \sum_{r=0}^n \oplus \bigwedge^r V = \bigoplus_{r=0}^n \bigwedge^r V.$$

To simplify the notation we sometimes write *simply*

$$\bigwedge V = \mathbb{R} + V + \bigwedge^2 V + \cdots + \bigwedge^{n-1} V + \bigwedge^n V.$$

As it is well known the set of multivectors over V has a natural structure of vector space over \mathbb{R} and we have

$$\begin{aligned} \dim \bigwedge V &= \dim \mathbb{R} + \dim V + \dim \bigwedge^2 V + \cdots + \dim \bigwedge^{n-1} V + \dim \bigwedge^n V \\ &= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n. \end{aligned} \quad (3)$$

An element of $\bigwedge V$ will be called a multivector over V . If $X \in \bigwedge V$ we write:

$$X = X^0 + X^1 + X^2 + \cdots + X^{n-1} + X^n.$$

In what follows we shall need also the vector space $\bigwedge V^* = \bigoplus_{r=0}^n \bigwedge^r V^*$. An element of $\bigwedge V^*$ will be called a *multiform over V* . As in the case of

multivectors we simplify our notation and as in the case of multivectors we simply write:

$$\bigwedge V^* = \mathbb{R} + V^* + \bigwedge^2 V^* + \cdots + \bigwedge^{n-1} V^* + \bigwedge^n V^*, \quad (4)$$

and if $\Phi \in \bigwedge V^*$ we write

$$\Phi = \Phi_0 + \Phi_1 + \Phi_2 + \cdots + \Phi_{n-1} + \Phi_n. \quad (5)$$

Of course, $\bigwedge V^*$ has a natural structure of real vector space over \mathbb{R} . We have,

$$\begin{aligned} \dim \bigwedge V^* &= \dim \mathbb{R} + \dim V^* + \dim \bigwedge^2 V^* + \cdots + \dim \bigwedge^{n-1} V^* + \dim \bigwedge^n V^* \\ &= \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n. \end{aligned} \quad (6)$$

We recall that $\bigwedge^k V$ is also called the homogeneous multivector space (of degree k), and to $\bigwedge^k V^*$ the homogeneous multiform space (of degree k).

Let us take an integer number k with $0 \leq k \leq n$. The linear mappings

$$\bigwedge V \ni X \longmapsto \langle X \rangle^k \in \bigwedge V \text{ and } \bigwedge V^* \ni \Phi \longmapsto \langle \Phi \rangle_k \in \bigwedge V^*$$

such that if $X = X^0 + X^1 + \cdots + X^n$ and $\Phi = \Phi_0 + \Phi_1 + \cdots + \Phi_n$, then

$$\langle X \rangle^k = X^k \text{ and } \langle \Phi \rangle_k = \Phi_k \quad (7)$$

are called the *k-part operator (for multivectors)* and the *k-part operator (for multiforms)*, respectively. $\langle X \rangle^k$ is read as the *k-part of X* and $\langle \Phi \rangle_k$ is read as the *k-part of Φ*.

It should be evident that for all $X \in \bigwedge V$ and $\Phi \in \bigwedge V^*$:

$$X = \sum_{k=0}^n \langle X \rangle^k, \quad (8)$$

$$\Phi = \sum_{k=0}^n \langle \Phi \rangle_k. \quad (9)$$

The linear mappings

$$\bigwedge V \ni X \longmapsto \widehat{X} \in \bigwedge V \text{ and } \bigwedge V^* \ni \Phi \longmapsto \widehat{\Phi} \in \bigwedge V^*$$

such that

$$\langle \widehat{X} \rangle^k = (-1)^k \langle X \rangle^k \text{ and } \langle \widehat{\Phi} \rangle_k = (-1)^k \langle \Phi \rangle_k \quad (10)$$

are called the *grade involution operator (for multivectors)* and the *grade involution operator (for multiforms)*, respectively.

The linear mappings

$$\bigwedge V \ni X \mapsto \widetilde{X} \in \bigwedge V \text{ and } \bigwedge V^* \ni \Phi \mapsto \widetilde{\Phi} \in \bigwedge V^*$$

such that

$$\langle \widetilde{X} \rangle^k = (-1)^{\frac{1}{2}k(k-1)} \langle X \rangle^k \text{ and } \langle \widetilde{\Phi} \rangle_k = (-1)^{\frac{1}{2}k(k-1)} \langle \Phi \rangle_k \quad (11)$$

are called the *reversion operator (for multivectors)* and the *reversion operator (for multiforms)*, respectively.

Both of $\bigwedge V$ and $\bigwedge V^*$ endowed with the exterior product \wedge (of multivectors and multiforms!) are *associative algebras*, i.e., the *exterior algebra of multivectors* and the *exterior algebra of multiforms*, respectively.

3 Duality Scalar Product

The duality scalar product of a multiform Φ with a multivector X is the scalar $\langle \Phi, X \rangle$ (i.e., the real number) defined by the following axioms

- For all $\alpha, \beta \in \mathbb{R}$:

$$\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = \alpha \beta. \quad (12)$$

- For all $\Phi_p \in \bigwedge^p V^*$ and $X^p \in \bigwedge^p V$ (with $1 \leq p \leq n$):

$$\langle \Phi_p, X^p \rangle = \langle X^p, \Phi_p \rangle = \frac{1}{p!} \Phi_p(e_{j_1}, \dots, e_{j_p}) X^p(\varepsilon^{j_1}, \dots, \varepsilon^{j_p}), \quad (13)$$

where $\{e_j, \varepsilon^j\}$ is any pair of dual bases over V .

- For all $\Phi \in \bigwedge V^*$ and $X \in \bigwedge V$: if $\Phi = \Phi_0 + \Phi_1 + \dots + \Phi_n$ and $X = X^0 + X^1 + \dots + X^n$, then

$$\langle \Phi, X \rangle = \langle X, \Phi \rangle = \sum_{p=0}^n \langle \Phi_p, X^p \rangle. \quad (14)$$

We emphasize that the scalar $\Phi_p(e_{j_1}, \dots, e_{j_p})X^p(\varepsilon^{j_1}, \dots, \varepsilon^{j_p})$ has frame independent character, i.e., it does not depend on the pair of dual bases $\{e_j, \varepsilon^j\}$ used for calculating it, since Φ_p and X^p are p -linear mappings.

Note that for all $\omega \in V^*$ and $v \in V$ it holds

$$\langle v, \omega \rangle = \langle \omega, v \rangle = \omega(v). \quad (15)$$

We present two noticeable properties for the duality scalar product between p -forms and p -vectors

- For all $\Phi_p \in \bigwedge^p V^*$, and $v_1, \dots, v_p \in V$:

$$\langle \Phi_p, v_1 \wedge \dots \wedge v_p \rangle = \langle v_1 \wedge \dots \wedge v_p, \Phi_p \rangle = \Phi_p(v_1, \dots, v_p). \quad (16)$$

- For all $\omega^1, \dots, \omega^p \in V^*$ and $v_1, \dots, v_p \in V$:

$$\langle \omega^1 \wedge \dots \wedge \omega^p, v_1 \wedge \dots \wedge v_p \rangle = \det \begin{pmatrix} \omega^1(v_1) & \dots & \omega^1(v_p) \\ \vdots & & \vdots \\ \omega^p(v_1) & \dots & \omega^p(v_p) \end{pmatrix}. \quad (17)$$

The basic properties for the duality scalar product are the non-degeneracy and the distributive laws on the left and on the right with respect to addition of either multiforms or multivectors, i.e.,

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$$\begin{aligned} \langle \Phi, X \rangle &= 0, \text{ for all } \Phi \implies X = 0, \\ \langle \Phi, X \rangle &= 0, \text{ for all } X \implies \Phi = 0, \end{aligned} \quad (18)$$

and,

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$$\begin{aligned} \langle \Phi + \Psi, X \rangle &= \langle \Phi, X \rangle + \langle \Psi, X \rangle, \\ \langle \Phi, X + Y \rangle &= \langle \Phi, X \rangle + \langle \Phi, Y \rangle. \end{aligned} \quad (19)$$

4 Duality Contracted Products

4.1 Left Contracted Product

The left contracted product of a multiform Φ with a multivector X (or, a multivector X with a multiform Φ) is the multivector $\langle \Phi, X|$ (respectively, the multiform $\langle X, \Phi|$) defined by the following axioms:

- For all $\Phi_p \in \bigwedge^p V^*$ and $X^p \in \bigwedge^p V$ with $0 \leq p \leq n$:

$$\langle \Phi_p, X^p| = \langle X^p, \Phi_p| = \langle \tilde{\Phi}_p, X^p \rangle = \langle \Phi_p, \tilde{X}^p \rangle. \quad (20)$$

- For all $\Phi_p \in \bigwedge^p V^*$ and $X^q \in \bigwedge^q V$ (or $X^p \in \bigwedge^p V$ and $\Phi_q \in \bigwedge^q V^*$) with $0 \leq p < q \leq n$:

$$\langle \Phi_p, X^q| = \frac{1}{(q-p)!} \langle \tilde{\Phi}_p \wedge \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{q-p}}, X^q \rangle e_{j_1} \wedge \dots \wedge e_{j_{q-p}}, \quad (21)$$

$$\langle X^p, \Phi_q| = \frac{1}{(q-p)!} \langle \tilde{X}^p \wedge e_{j_1} \wedge \dots \wedge e_{j_{q-p}}, \Phi_q \rangle \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{q-p}}, \quad (22)$$

where $\{e_j, \varepsilon^j\}$ is any pair of dual bases for V and V^* .

- For all $\Phi \in \bigwedge V^*$ and $X \in \bigwedge V$: if $\Phi = \Phi_0 + \Phi_1 + \dots + \Phi_n$ and $X = X^0 + X^1 + \dots + X^n$, then

$$\langle \Phi, X| = \sum_{k=0}^n \sum_{j=0}^{n-k} \langle \Phi_j, X^{k+j}|, \quad (23)$$

$$\langle X, \Phi| = \sum_{k=0}^n \sum_{j=0}^{n-k} \langle X^j, \Phi_{k+j}|. \quad (24)$$

Note that the $(q-p)$ -vector $\langle \Phi_p, X^q|$ and the $(q-p)$ -form $\langle X^p, \Phi_q|$ have frame independent character, i.e., they do not depend on the pair of frames $\{e_j, \varepsilon^j\}$ chosen for calculating them.

The left contracted product has the following basic properties:

- Let us take $\Phi_p \in \bigwedge^p V^*$ and $X^q \in \bigwedge^q V$ with $0 \leq p \leq q \leq n$. For all $\Psi_{q-p} \in \bigwedge^{q-p} V^*$, it holds

$$\langle \langle \Phi_p, X^q |, \Psi_{q-p} \rangle = \langle X^q, \tilde{\Phi}_p \wedge \Psi_{q-p} \rangle. \quad (25)$$

- Let us take $X^p \in \bigwedge^p V$ and $\Phi_q \in \bigwedge^q V^*$ with $0 \leq p \leq q \leq n$. For all $Y^{q-p} \in \bigwedge^{q-p} V$, it holds

$$\langle \langle X^p, \Phi_q |, Y^{q-p} \rangle = \langle \Phi_q, \tilde{X}^p \wedge Y^{q-p} \rangle. \quad (26)$$

- For all $X \in \bigwedge V$ and $\Phi, \Psi \in \bigwedge V^*$:

$$\langle \langle \Phi, X |, \Psi \rangle = \langle X, \tilde{\Phi} \wedge \Psi \rangle. \quad (27)$$

- For all $\Phi \in \bigwedge V^*$ and $X, Y \in \bigwedge V$:

$$\langle \langle X, \Phi |, Y \rangle = \langle \Phi, \tilde{X} \wedge Y \rangle. \quad (28)$$

The left contracted product satisfies the distributive laws on the left and on the right.

- For all $\Phi, \Psi \in \bigwedge V^*$ and $X, Y \in \bigwedge V$:

$$\begin{aligned} \langle (\Phi + \Psi), X | &= \langle \Phi, X | + \langle \Psi, X |, \\ \langle \Phi, (X + Y) | &= \langle \Phi, X | + \langle \Phi, Y |. \end{aligned} \quad (29)$$

- For all $X, Y \in \bigwedge V$ and $\Phi, \Psi \in \bigwedge V^*$:

$$\begin{aligned} \langle (X + Y), \Phi | &= \langle X, \Phi | + \langle Y, \Phi |, \\ \langle X, (\Phi + \Psi) | &= \langle X, \Phi | + \langle X, \Psi |. \end{aligned} \quad (30)$$

Proof. We present only the proof of the property given by Eq. (25), the other proofs being somewhat analogous.

First note that if $\Psi_{q-p} \in \bigwedge^{q-p} V^*$ and $\{e_j, \varepsilon^j\}$ is any pair of dual bases for V and V^* , we can write

$$\Psi_{q-p} = \frac{1}{(q-p)!} \langle \Psi_{q-p}, e_{j_1} \wedge \dots \wedge e_{j_{q-p}} \rangle \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{q-p}}.$$

Then, using the definition (21) and the above equation we have

$$\begin{aligned}
& \langle \langle \Phi_p, X^q |, \Psi_{q-p} \rangle \\
&= \frac{1}{(q-p)!} \left\langle \left\langle \tilde{\Phi}_p \wedge \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{q-p}}, X^q \right\rangle e_{j_1} \wedge \dots \wedge e_{j_{q-p}}, \Psi_{q-p} \right\rangle \\
&= \frac{1}{(q-p)!} \left\langle \tilde{\Phi}_p \wedge \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{q-p}}, X^q \right\rangle \langle e_{j_1} \wedge \dots \wedge e_{j_{q-p}}, \Psi_{q-p} \rangle \\
&= \left\langle X^q, \frac{1}{(q-p)!} \langle \Psi_{q-p}, e_{j_1} \wedge \dots \wedge e_{j_{q-p}} \rangle \tilde{\Phi}_p \wedge \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{q-p}} \right\rangle \\
&= \left\langle X^q, \tilde{\Phi}_p \wedge \frac{1}{(q-p)!} \langle \Psi_{q-p}, e_{j_1} \wedge \dots \wedge e_{j_{q-p}} \rangle \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{q-p}} \right\rangle \\
&= \left\langle X^q, \tilde{\Phi}_p \wedge \Psi_{q-p} \right\rangle,
\end{aligned}$$

and the result is proved. ■

4.2 Right Contracted Product

The right contracted product of a multiform Φ with a multivector X (or, a multivector X with a multiform Φ) is the multiform $|\Phi, X\rangle$ (respectively, the multivector $|X, \Phi\rangle$) defined by the following axioms:

- For $\Phi_p \in \bigwedge^p V^*$ and $X^p \in \bigwedge^p V$ with $n \geq p \geq 0$:

$$|\Phi_p, X^p\rangle = |X^p, \Phi_p\rangle = \langle \tilde{\Phi}_p, X^p \rangle = \langle \Phi_p, \tilde{X}^p \rangle. \quad (31)$$

- For $\Phi_p \in \bigwedge^p V^*$ and $X^q \in \bigwedge^q V$ (or $X^p \in \bigwedge^p V$ and $\Phi_q \in \bigwedge^q V^*$) with $n \geq p > q \geq 0$:

$$|\Phi_p, X^q\rangle = \frac{1}{(p-q)!} \langle \Phi_p, e_{j_1} \wedge \dots \wedge e_{j_{p-q}} \wedge \tilde{X}^q \rangle \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{p-q}}, \quad (32)$$

$$|X^p, \Phi_q\rangle = \frac{1}{(p-q)!} \langle X^p, \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{p-q}} \wedge \tilde{\Phi}_q \rangle e_{j_1} \wedge \dots \wedge e_{j_{p-q}}, \quad (33)$$

where $\{e_j, \varepsilon^j\}$ is any pair of dual bases for V and V^* .

- For all $\Phi \in \bigwedge V^*$ and $X \in \bigwedge V$: if $\Phi = \Phi_0 + \Phi_1 + \cdots + \Phi_n$ and $X = X^0 + X^1 + \cdots + X^n$, then

$$|\Phi, X\rangle = \sum_{k=0}^n \sum_{j=0}^{n-k} |\Phi_{k+j}, X^j\rangle, \quad (34)$$

$$|X, \Phi\rangle = \sum_{k=0}^n \sum_{j=0}^{n-k} |X^{k+j}, \Phi_j\rangle. \quad (35)$$

The right contracted product satisfies the following basic properties:

- Let us take $\Phi_p \in \bigwedge^p V^*$ and $X^q \in \bigwedge^q V$ with $n \geq p \geq q \geq 0$. For all $Y^{p-q} \in \bigwedge^{p-q} V$, it holds

$$\langle Y^{p-q}, |\Phi_p, X^q\rangle \rangle = \langle Y^{p-q} \wedge \tilde{X}^q, \Phi_p \rangle. \quad (36)$$

- Let us take $X^p \in \bigwedge^p V$ and $\Phi_q \in \bigwedge^q V^*$ with $n \geq p \geq q \geq 0$. For all $\Psi_{p-q} \in \bigwedge^{p-q} V^*$, it holds

$$\langle \Psi_{p-q}, |X^p, \Phi_q\rangle \rangle = \langle \Psi_{p-q} \wedge \tilde{\Phi}_q, X^p \rangle. \quad (37)$$

- For all $\Phi \in \bigwedge V^*$ and $X, Y \in \bigwedge V$:

$$\langle Y, |\Phi, X\rangle \rangle = \langle Y \wedge \tilde{X}, \Phi \rangle. \quad (38)$$

- For all $X \in \bigwedge V$ and $\Phi, \Psi \in \bigwedge V^*$:

$$\langle \Psi, |X, \Phi\rangle \rangle = \langle \Psi \wedge \tilde{\Phi}, X \rangle. \quad (39)$$

The right contracted product satisfies also the following distributive laws:

- For all $\Phi, \Psi \in \bigwedge V^*$ and $X, Y \in \bigwedge V$:

$$\begin{aligned} |(\Phi + \Psi), X\rangle &= |\Phi, X\rangle + |\Psi, X\rangle, \\ |\Phi, (X + Y)\rangle &= |\Phi, X\rangle + |\Phi, Y\rangle. \end{aligned} \quad (40)$$

- For all $X, Y \in \bigwedge V$ and $\Phi, \Psi \in \bigwedge V^*$:

$$\begin{aligned} |(X + Y), \Phi\rangle &= |X, \Phi\rangle + |Y, \Phi\rangle, \\ |X, (\Phi + \Psi)\rangle &= |X, \Phi\rangle + |X, \Psi\rangle. \end{aligned} \quad (41)$$

We present two noticeable properties relating the left and right contracted products.

- For all $\Phi_p \in \bigwedge^p V^*$ and $X^q \in \bigwedge^q V$ with $p \leq q$:

$$\langle \Phi_p, X^q | = (-1)^{p(q-p)} |X^q, \Phi_p\rangle. \quad (42)$$

- For all $X^p \in \bigwedge^p V$ and $\Phi_q \in \bigwedge^q V^*$ with $p \leq q$:

$$\langle X^p, \Phi_q | = (-1)^{p(q-p)} |\Phi_q, X^p\rangle. \quad (43)$$

Proof. We present only the proof of the property given by Eq. (42), the proofs of the other properties are analogous. Let $\Phi_p \in \bigwedge^p V^*$ and $X^q \in \bigwedge^q V$ with $p \leq q$, then

$$\begin{aligned} \langle \Phi_p, X^q | &= \frac{1}{(q-p)!} \left\langle \tilde{\Phi}_p \wedge \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{q-p}}, X^q \right\rangle e_{j_1} \wedge \dots \wedge e_{j_{q-p}} \\ &= \frac{1}{(q-p)!} \left\langle (-1)^{p(q-p)} \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{q-p}} \wedge \tilde{\Phi}_p, X^q \right\rangle e_{j_1} \wedge \dots \wedge e_{j_{q-p}} \\ &= (-1)^{p(q-p)} \frac{1}{(q-p)!} \left\langle X^q, \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_{q-p}} \wedge \tilde{\Phi}_p \right\rangle e_{j_1} \wedge \dots \wedge e_{j_{q-p}} \\ &= (-1)^{p(q-p)} |X^q, \Phi_p\rangle, \end{aligned}$$

and the result is proved. ■

5 Extensors

Let $\bigwedge_1^\diamond V, \dots$ and $\bigwedge_k^\diamond V$ be k subspaces of $\bigwedge V$ such that each of them is any sum of homogeneous subspaces of $\bigwedge V$, and let $\bigwedge_1^\diamond V^*, \dots$ and $\bigwedge_l^\diamond V^*$ be l subspaces of $\bigwedge V^*$ such that each of them is any sum of homogeneous subspaces of $\bigwedge V^*$.

If $\bigwedge^\diamond V$ is any sum of homogeneous subspaces of $\bigwedge V$, a multilinear mapping

$$\begin{aligned} \underbrace{\bigwedge_1^\diamond V \times \dots \times \bigwedge_k^\diamond V}_{k\text{-copies}} \times \underbrace{\bigwedge_1^\diamond V^* \times \dots \times \bigwedge_l^\diamond V^*}_{l\text{-copies}} &\ni (X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) \\ &\longmapsto \tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) \in \bigwedge^\diamond V \end{aligned} \quad (44)$$

is called a k multivector and l multiform variables multivector extensor over V .

If $\bigwedge^\diamond V^*$ is any sum of homogeneous subspaces of $\bigwedge V^*$, a multilinear mapping

$$\underbrace{\bigwedge_1^\diamond V \times \cdots \times \bigwedge_k^\diamond V}_{k\text{-copies}} \times \underbrace{\bigwedge_1^\diamond V^* \times \cdots \times \bigwedge_l^\diamond V^*}_{l\text{-copies}} \ni (X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) \longmapsto \tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) \in \bigwedge^\diamond V^* \quad (45)$$

is called a k multivector and l multiform variables multiform extensor over V .

The set of all the k multivector and l multiform variables multivector extensors over V has a natural structure of real vector space, and will be denoted by the highly suggestive notation $ext(\bigwedge_1^\diamond V, \dots, \bigwedge_k^\diamond V, \bigwedge_1^\diamond V^*, \dots, \bigwedge_l^\diamond V^*; \bigwedge^\diamond V)$. When no confusion arises, we use the more simple notation $ext_k^l(V)$ for that space.

We obviously have that:

$$\dim ext_k^l(V) = \dim \bigwedge_1^\diamond V \dots \dim \bigwedge_k^\diamond V \dim \bigwedge_1^\diamond V^* \dots \dim \bigwedge_l^\diamond V^* \dim \bigwedge^\diamond V. \quad (46)$$

The set of all the k multivector and l multiform variables multiform extensors over V has also a natural structure of real vector space, and will be denoted by $ext(\bigwedge_1^\diamond V, \dots, \bigwedge_k^\diamond V, \bigwedge_1^\diamond V^*, \dots, \bigwedge_l^\diamond V^*; \bigwedge^\diamond V^*)$, and when no confusion arises, we use the simple notation $ext_k^{*l}(V)$ for this space. Also, we have,

$$\dim ext_k^{*l}(V) = \dim \bigwedge_1^\diamond V \dots \dim \bigwedge_k^\diamond V \dim \bigwedge_1^\diamond V^* \dots \dim \bigwedge_l^\diamond V^* \dim \bigwedge^\diamond V^*. \quad (47)$$

6 Algebra of Extensors

6.1 Exterior Product of Extensors

We define the exterior product of $\tau \in ext_k^l(V)$ and $\sigma \in ext_r^s(V)$ (or, $\tau \in ext_k^{*l}(V)$ and $\sigma \in ext_r^{*s}(V)$) as $\tau \wedge \sigma \in ext_{k+r}^{l+s}(V)$ (respectively,

$\tau \wedge \sigma \in \overset{*}{ext}_{k+r}^{l+s}(V)$ given by

$$\begin{aligned} \tau \wedge \sigma & (X_1, \dots, X_k, Y_1, \dots, Y_r, \Phi^1, \dots, \Phi^l, \Psi^1, \dots, \Psi^s) \\ & = \tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) \wedge \sigma(Y_1, \dots, Y_r, \Psi^1, \dots, \Psi^s). \end{aligned} \quad (48)$$

Note that on the right side appears an exterior product of multivectors (respectively, an exterior product of multiforms).

The duality scalar product of a multiform extensor $\tau \in \overset{*}{ext}_k^l(V)$ with a multivector extensor $\sigma \in \overset{s}{ext}_r(V)$ is the *scalar* extensor $\langle \tau, \sigma \rangle \in \overset{*}{ext}_{k+r}^{l+s}(V)$ defined by

$$\begin{aligned} \langle \tau, \sigma \rangle & (X_1, \dots, X_k, Y_1, \dots, Y_r, \Phi^1, \dots, \Phi^l, \Psi^1, \dots, \Psi^s) \\ & = \langle \tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l), \sigma(Y_1, \dots, Y_r, \Psi^1, \dots, \Psi^s) \rangle. \end{aligned} \quad (49)$$

The duality left contracted product of a multiform extensor $\tau \in \overset{*}{ext}_k^l(V)$ with a multivector extensor $\sigma \in \overset{s}{ext}_r(V)$ (or, a multivector extensor $\tau \in \overset{l}{ext}_k(V)$ with a multiform extensor $\sigma \in \overset{*}{ext}_r^s(V)$) is the multivector extensor $\langle \tau, \sigma | \in \overset{l+s}{ext}_{k+r}(V)$ (respectively, the multiform extensor $\langle \tau, \sigma | \in \overset{*}{ext}_{k+r}^{l+s}(V)$) defined by

$$\begin{aligned} \langle \tau, \sigma | & (X_1, \dots, X_k, Y_1, \dots, Y_r, \Phi^1, \dots, \Phi^l, \Psi^1, \dots, \Psi^s) \\ & = \langle \tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l), \sigma(Y_1, \dots, Y_r, \Psi^1, \dots, \Psi^s) \rangle. \end{aligned} \quad (50)$$

The duality right contracted product of a multiform extensor $\tau \in \overset{*}{ext}_k^l(V)$ with a multivector extensor $\sigma \in \overset{s}{ext}_r(V)$ (or, a multivector extensor $\tau \in \overset{l}{ext}_k(V)$ with a multiform extensor $\sigma \in \overset{*}{ext}_r^s(V)$) is the multiform extensor $|\tau, \sigma \rangle \in \overset{*}{ext}_{k+r}^{l+s}(V)$ (respectively, the multivector extensor $|\tau, \sigma \rangle \in \overset{l+s}{ext}_{k+r}(V)$) defined by

$$\begin{aligned} |\tau, \sigma \rangle & (X_1, \dots, X_k, Y_1, \dots, Y_r, \Phi^1, \dots, \Phi^l, \Psi^1, \dots, \Psi^s) \\ & = |\tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l), \sigma(Y_1, \dots, Y_r, \Psi^1, \dots, \Psi^s) \rangle. \end{aligned} \quad (51)$$

6.2 Duality Adjoint of Extensors

Let $\bigwedge^\diamond V$ be any sum of homogeneous subspaces of $\bigwedge V$. There exist μ integer numbers p_1, \dots, p_μ with $0 \leq p_1 < \dots < p_\mu \leq n$ such that $\bigwedge^\diamond V = \bigwedge^{p_1} V + \dots + \bigwedge^{p_\mu} V$. Analogously, if $\bigwedge^\diamond V^*$ is any sum of homogeneous subspaces of $\bigwedge V^*$, then there exist ν integer numbers q_1, \dots, q_ν with $0 \leq q_1 < \dots < q_\nu \leq n$ such that $\bigwedge^\diamond V^* = \bigwedge^{q_1} V^* + \dots + \bigwedge^{q_\nu} V^*$.

The linear mappings

$$\bigwedge V \ni X \longmapsto \langle X \rangle \bigwedge^\diamond V \in \bigwedge V \text{ and } \bigwedge V^* \ni \Phi \longmapsto \langle \Phi \rangle \bigwedge^\diamond_{V^*} \in \bigwedge V^*$$

such that if $\bigwedge^\diamond V = \bigwedge^{p_1} V + \dots + \bigwedge^{p_\mu} V$ and $\bigwedge^\diamond V^* = \bigwedge^{q_1} V^* + \dots + \bigwedge^{q_\nu} V^*$, then

$$\langle X \rangle \bigwedge^\diamond V = \langle X \rangle^{p_1} + \dots + \langle X \rangle^{p_\mu} \text{ and } \langle \Phi \rangle \bigwedge^\diamond_{V^*} = \langle \Phi \rangle_{q_1} + \dots + \langle \Phi \rangle_{q_\nu} \quad (52)$$

are called the $\bigwedge^\diamond V$ -part operator (for multivectors) and $\bigwedge^\diamond V^*$ -part operator (for multiforms), respectively.

It should be evident that for all $X \in \bigwedge V$ and $\Phi \in \bigwedge V^*$:

$$\langle X \rangle \bigwedge^k V = \langle X \rangle^k, \quad (53)$$

$$\langle \Phi \rangle \bigwedge^k_{V^*} = \langle \Phi \rangle_k. \quad (54)$$

Thus, $\bigwedge^\diamond V$ -part operator and $\bigwedge^\diamond V^*$ -part operator are the generalizations of $\langle \rangle^k$ and $\langle \rangle_k$.

Let τ be a multivector extensor of either one multivector variable or one multiform variable.

If $\tau \in \text{ext}(\bigwedge_1^\diamond V; \bigwedge_2^\diamond V)$ (or, $\tau \in \text{ext}(\bigwedge_3^\diamond V^*; \bigwedge_4^\diamond V)$), then $\tau^\Delta \in \text{ext}(\bigwedge_2^\diamond V^*; \bigwedge_1^\diamond V^*)$ (respectively, $\tau^\Delta \in \text{ext}(\bigwedge_4^\diamond V^*; \bigwedge_3^\diamond V)$) defined by

$$\begin{aligned} \tau^\Delta(\Phi) &= \left\langle \Phi, \tau(\langle 1 \rangle \bigwedge_1^\diamond V) \right\rangle \\ &+ \sum_{k=1}^n \frac{1}{k!} \left\langle \Phi, \tau(\langle e_{j_1} \wedge \dots \wedge e_{j_k} \rangle \bigwedge_1^\diamond V) \right\rangle \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_k} \end{aligned} \quad (55)$$

for each $\Phi \in \bigwedge_2^\diamond V^*$ (respectively, for each $\Phi \in \bigwedge_4^\diamond V^*$)

$$\begin{aligned}\tau^\Delta(\Phi) &= \left\langle \Phi, \tau(\langle 1 \rangle \wedge_3^\diamond V^*) \right\rangle \\ &+ \sum_{k=1}^n \frac{1}{k!} \left\langle \Phi, \tau(\langle \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_k} \rangle \wedge_3^\diamond V^*) \right\rangle e_{j_1} \wedge \dots \wedge e_{j_k}\end{aligned}\quad (56)$$

is called the *duality adjoint* of τ .

The basic properties of the adjoint of multivector extensors are:

- Let us take $\tau \in \text{ext}(\wedge_1^\diamond V; \wedge_2^\diamond V)$. For all $X \in \wedge_1^\diamond V$ and $\Phi \in \wedge_2^\diamond V^*$, it holds

$$\langle \tau(X), \Phi \rangle = \langle X, \tau^\Delta(\Phi) \rangle. \quad (57)$$

- Let us take $\tau \in \text{ext}(\wedge_3^\diamond V^*; \wedge_4^\diamond V)$. For all $\Phi \in \wedge_3^\diamond V^*$ and $\Psi \in \wedge_4^\diamond V^*$, it holds

$$\langle \tau(\Phi), \Psi \rangle = \langle \Phi, \tau^\Delta(\Psi) \rangle. \quad (58)$$

Proof. We present only the proof of the property given by Eq. (57), the other is analogous. First, observe that if $X \in \wedge_1^\diamond V$, then there exists μ integer numbers p_1, \dots, p_μ with $0 \leq p_1 < \dots < p_\mu \leq n$ such that $X = X^{p_1} + \dots + X^{p_\mu}$ with $X^{p_i} \in \wedge^{p_i} V$, where the $\wedge^{p_i} V$ are homogeneous subspace of $\wedge V$, thus if $\tau \in \text{ext}(\wedge_1^\diamond V; \wedge_2^\diamond V)$, we have that

$$\tau : \wedge_1^\diamond V \rightarrow \wedge_2^\diamond V \quad \text{or} \quad \tau : \wedge^{p_1} V + \dots + \wedge^{p_\mu} V \rightarrow \wedge^{q_1} V + \dots + \wedge^{q_\mu} V,$$

where $\wedge_1^\diamond V = \wedge^{p_1} V + \dots + \wedge^{p_\mu} V$ and $\wedge_2^\diamond V = \wedge^{q_1} V + \dots + \wedge^{q_\mu} V$, such that

$$\tau(X) = \tau(X^{p_1} + \dots + X^{p_\mu}) = \tau(X^{p_1}) + \dots + \tau(X^{p_\mu}) \in \wedge^{q_1} V + \dots + \wedge^{q_\mu} V,$$

thus, we can put

$$\tau(X^{p_i}) \in \wedge^{q_i} V \quad \text{or} \quad \tau \Big|_{\wedge^{p_i} V} \equiv \tau_{p_i} \in \text{ext}(\wedge^{p_i} V; \wedge^{q_i} V).$$

Now, with the above observation, we have

$$\begin{aligned}\langle \tau(X), \Phi \rangle &= \langle \tau(X^{p_1} + \dots + X^{p_\mu}), \Phi_{q_1} + \dots + \Phi_{q_\mu} \rangle \\ &= \langle \tau_{p_1}(X^{p_1}) + \dots + \tau_{p_\mu}(X^{p_\mu}), \Phi_{q_1} + \dots + \Phi_{q_\mu} \rangle \\ &= \sum_{i,j} \langle \tau_{p_i}(X^{p_i}), \Phi_{q_j} \rangle,\end{aligned}\quad (59)$$

but, by the definition (14) we have that

$$\langle \tau_{p_i}(X^{p_i}), \Phi^{q_j} \rangle \begin{cases} = 0 & \text{if } p_i \neq q_j \\ \neq 0 & \text{if } p_i = q_j = s_l \end{cases},$$

and from Eq. (59), we can write

$$\langle \tau(X), \Phi \rangle = \langle \tau_{s_1}(X^{s_1}), \Phi^{s_1} \rangle + \dots + \langle \tau_{s_\mu}(X^{s_\mu}), \Phi^{s_\mu} \rangle. \quad (60)$$

Now, if we see $\langle \tau_{s_l}(X^{s_l}), \Phi^{s_l} \rangle$ as a scalar product of s_l -vectors, then we have

$$\langle \tau_{s_l}(X^{s_l}), \Phi^{s_l} \rangle = \langle X^{s_l}, \tau_{s_l}^\Delta \Phi^{s_l} \rangle, \quad (61)$$

and from Eqs. (60), (61), and taking into account Eq. (59) we have

$$\begin{aligned} \langle \tau(X), \Phi \rangle &= \langle \tau_{s_1}(X^{s_1}), \Phi^{s_1} \rangle + \dots + \langle \tau_{s_\mu}(X^{s_\mu}), \Phi^{s_\mu} \rangle \\ &= \langle X^{s_1}, \tau_{s_1}^\Delta \Phi^{s_1} \rangle + \dots + \langle X^{s_\mu}, \tau_{s_\mu}^\Delta \Phi^{s_\mu} \rangle \\ &= \langle X^{s_1} + \dots + X^{s_\mu}, \tau_{s_1}^\Delta \Phi^{s_1} + \dots + \tau_{s_\mu}^\Delta \Phi^{s_\mu} \rangle \\ &= \langle X, \tau^\Delta \Phi \rangle, \end{aligned}$$

and the result is proved. ■

Let σ be a multiform extensor of either one multivector variable or one multiform variable. Then, if $\sigma \in \text{ext}(\bigwedge_1^\diamond V; \bigwedge_2^\diamond V^*)$ (or, $\sigma \in \text{ext}(\bigwedge_3^\diamond V^*; \bigwedge_4^\diamond V^*)$), then $\sigma^\Delta \in \text{ext}(\bigwedge_2^\diamond V; \bigwedge_1^\diamond V^*)$ (respectively, $\sigma^\Delta \in \text{ext}(\bigwedge_4^\diamond V; \bigwedge_3^\diamond V)$) defined by

$$\begin{aligned} \sigma^\Delta(X) &= \left\langle X, \sigma(\langle 1 \rangle \bigwedge_1^\diamond V) \right\rangle \\ &+ \sum_{k=1}^n \frac{1}{k!} \left\langle X, \sigma(\langle e_{j_1} \wedge \dots \wedge e_{j_k} \rangle \bigwedge_1^\diamond V) \right\rangle \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_k} \end{aligned} \quad (62)$$

for each $X \in \bigwedge_2^\diamond V$ (respectively,

$$\begin{aligned} \sigma^\Delta(X) &= \left\langle X, \sigma(\langle 1 \rangle \bigwedge_3^\diamond V^*) \right\rangle \\ &+ \sum_{k=1}^n \frac{1}{k!} \left\langle X, \sigma(\langle \varepsilon^{j_1} \wedge \dots \wedge \varepsilon^{j_k} \rangle \bigwedge_3^\diamond V^*) \right\rangle e_{j_1} \wedge \dots \wedge e_{j_k} \end{aligned} \quad (63)$$

for each $X \in \bigwedge_4^\diamond V$) is called the *duality adjoint* of σ .

The basic properties for the adjoint of multiform extensors are:

- Let us take $\sigma \in \text{ext}(\bigwedge_1^\diamond V; \bigwedge_2^\diamond V^*)$. For all $X \in \bigwedge_1^\diamond V$ and $Y \in \bigwedge_2^\diamond V$, it holds

$$\langle \sigma(X), Y \rangle = \langle X, \sigma^\Delta(Y) \rangle. \quad (64)$$

- Let us take $\sigma \in \text{ext}(\bigwedge_3^\diamond V^*; \bigwedge_4^\diamond V^*)$. For all $\Phi \in \bigwedge_3^\diamond V^*$ and $X \in \bigwedge_4^\diamond V$, it holds

$$\langle \sigma(\Phi), X \rangle = \langle \Phi, \sigma^\Delta(X) \rangle. \quad (65)$$

The linear mapping $(\)^\Delta$ will be called the *duality adjoint operator*.

7 Extension Procedure for Operators

Let λ be a linear operator on V , i.e., a linear map $V \ni v \mapsto \lambda(v) \in V$. It can be extended in such a way as to give a linear operator on $\bigwedge V$, namely $\underline{\lambda}$, which is defined by

$$\bigwedge V \ni X \mapsto \underline{\lambda}(X) \in \bigwedge V$$

such that

$$\underline{\lambda}(X) = \langle 1, X \rangle + \sum_{k=1}^n \frac{1}{k!} \langle \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_k}, X \rangle \lambda(e_{j_1}) \wedge \cdots \wedge \lambda(e_{j_k}), \quad (66)$$

where $\{e_j, \varepsilon^j\}$ is any pair of dual bases for V and V^* .

We emphasize that $\underline{\lambda}$ is a well-defined linear operator on $\bigwedge V$. Note that each k -vector $\langle \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_k}, X \rangle \lambda(e_{j_1}) \wedge \cdots \wedge \lambda(e_{j_k})$ with $1 \leq k \leq n$ does not depend on the choice of $\{e_j, \varepsilon^j\}$, and the linearity follows just from the linearity of the duality scalar product. We call $\underline{\lambda}$ the *extended* of λ (to multivector operator).

The extended of a vector operator λ has the following basic properties:

- $\underline{\lambda}$ is grade-preserving, i.e.,

$$\text{if } X \in \bigwedge^k V, \text{ then } \underline{\lambda}(X) \in \bigwedge^k V. \quad (67)$$

- For all $\alpha \in \mathbb{R}$, $v \in V$, and $X, Y \in \bigwedge V$:

$$\underline{\lambda}(\alpha) = \alpha, \quad (68)$$

$$\underline{\lambda}(v) = \lambda(v), \quad (69)$$

$$\underline{\lambda}(X \wedge Y) = \underline{\lambda}(X) \wedge \underline{\lambda}(Y). \quad (70)$$

We observe that the four basic properties as given by Eq.(67), Eq.(68), Eq.(69) and Eq.(70) are completely equivalent to the extension procedure of a vector operator.

Let λ be a linear operator on V^* , i.e., a linear map $V^* \ni \omega \longmapsto \lambda(\omega) \in V^*$. It is possible to extend λ in such a way to get a linear operator on $\bigwedge V^*$, namely the operator $\underline{\lambda}$, defined by

$$\bigwedge V^* \ni \Phi \longmapsto \underline{\lambda}(\Phi) \in \bigwedge V^*,$$

such that

$$\underline{\lambda}(\Phi) = \langle 1, \Phi \rangle + \sum_{k=1}^n \frac{1}{k!} \langle e_{j_1} \wedge \cdots \wedge e_{j_k}, \Phi \rangle \lambda(\varepsilon^{j_1}) \wedge \cdots \wedge \lambda(\varepsilon^{j_k}), \quad (71)$$

where $\{e_j, \varepsilon^j\}$ is any pair of dual bases for V and V^* .

We emphasize that $\underline{\lambda}$ is a well-defined linear operator on $\bigwedge V^*$. We call $\underline{\lambda}$ the *extended* of λ (to multiforms).

The extended of a form operator λ has the following basic properties.

- $\underline{\lambda}$ is grade-preserving, i.e.,

$$\text{if } \Phi \in \bigwedge^k V^*, \text{ then } \underline{\lambda}(\Phi) \in \bigwedge^k V^*. \quad (72)$$

- For all $\alpha \in \mathbb{R}$, $\omega \in V^*$, and $\Phi, \Psi \in \bigwedge V^*$:

$$\underline{\lambda}(\alpha) = \alpha, \quad (73)$$

$$\underline{\lambda}(\omega) = \lambda(\omega), \quad (74)$$

$$\underline{\lambda}(\Phi \wedge \Psi) = \underline{\lambda}(\Phi) \wedge \underline{\lambda}(\Psi). \quad (75)$$

The four basic properties given by Eq.(72), Eq.(73) Eq.(74) and Eq.(75) are logically equivalent to the extension procedure of a form operator.

There exists a relationship between the extension procedure of a vector operator and the extension procedure of a form operator.

Let us take a vector operator (or, a form operator) λ . As we can see, the duality adjoint of λ is just a form operator (respectively, a vector operator), and the duality adjoint of $\underline{\lambda}$ is just a multiform operator (respectively, a multivector operator). It holds that the duality adjoint of the extended of λ is equal to the extended of the duality adjoint of λ , i.e.,

$$(\underline{\lambda})^\Delta = \underline{(\lambda^\Delta)}. \quad (76)$$

Thus, it is possible to use the more simple notation $\underline{\lambda}^\Delta$ to mean either $(\underline{\lambda})^\Delta$ or $(\underline{\lambda}^\Delta)$.

We present some properties for the extended of an invertible vector operator λ .

- For all $\Phi \in \bigwedge V^*$, and $X \in \bigwedge V$:

$$\underline{\lambda} \langle \Phi, X \rangle = \langle \underline{\lambda}^{-\Delta}(\Phi), \underline{\lambda}(X) \rangle, \quad (77)$$

$$\underline{\lambda} \langle \Phi, X | = \langle \underline{\lambda}^{-\Delta}(\Phi), \underline{\lambda}(X) |, \quad (78)$$

$$\underline{\lambda} |X, \Phi \rangle = | \underline{\lambda}(X), \underline{\lambda}^{-\Delta}(\Phi) \rangle. \quad (79)$$

We present only the proof for the property given by Eq.(78), the other proofs are analogous.

Proof. Let us take $X \in \bigwedge V$ and $\Phi, \Psi \in \bigwedge V^*$. A straightforward calculation, using Eq.(57), Eq.(27), Eq.(75) and Eq.(72), yields

$$\begin{aligned} \langle \underline{\lambda} \langle \Phi, X |, \Psi \rangle &= \langle \langle \Phi, X |, \underline{\lambda}^\Delta(\Psi) \rangle = \langle X, \widetilde{\Phi} \wedge \underline{\lambda}^\Delta(\Psi) \rangle \\ &= \langle X, \underline{\lambda}^\Delta(\underline{\lambda}^{-\Delta}(\widetilde{\Phi}) \wedge \Psi) \rangle = \langle \underline{\lambda}(X), \widetilde{\underline{\lambda}^{-\Delta}(\Phi)} \wedge \Psi \rangle \\ &= \langle \langle \underline{\lambda}^{-\Delta}(\Phi), \underline{\lambda}(X) |, \Psi \rangle, \end{aligned}$$

whence, by the non-degeneracy of duality scalar product the result follows. ■

We present now some properties for the extended of an invertible form operator λ .

- For all $\Phi \in \bigwedge V^*$, and $X \in \bigwedge V$:

$$\underline{\lambda} \langle \Phi, X \rangle = \langle \underline{\lambda}(\Phi), \underline{\lambda}^{-\Delta}(X) \rangle, \quad (80)$$

$$\underline{\lambda} \langle \Phi, X | = \langle \underline{\lambda}(\Phi), \underline{\lambda}^{-\Delta}(X) |, \quad (81)$$

$$\underline{\lambda} |X, \Phi \rangle = | \underline{\lambda}^{-\Delta}(X), \underline{\lambda}(\Phi) \rangle. \quad (82)$$

7.1 Action of Extended Operators on Extensors

Let λ be an invertible linear operator on V . As we saw above, the extended of λ , denoted $\underline{\lambda}$, maps multivectors over V into multivectors over

V . However, it is possible to extend the action of $\underline{\lambda}$ in such a way that $\underline{\lambda}$ maps multivector extensors over V into multivector extensors over V . We define the linear mapping

$$ext_k^l(V) \ni \tau \longmapsto \underline{\lambda}\tau \in ext_k^l(V)$$

such that

$$\underline{\lambda}\tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) = \underline{\lambda} \circ \tau(\underline{\lambda}^{-1}(X_1), \dots, \underline{\lambda}^{-1}(X_k), \underline{\lambda}^\Delta(\Phi^1), \dots, \underline{\lambda}^\Delta(\Phi^l)) \quad (83)$$

for each $X_1, \dots, X_k \in \bigwedge V$ and $\Phi^1, \dots, \Phi^l \in \bigwedge V^*$.

It that way $\underline{\lambda}$ can be thought as a linear multivector extensor operator.

For instance, for $\tau \in ext_1^0(V)$ the above definition means

$$\underline{\lambda}\tau = \underline{\lambda} \circ \tau \circ \underline{\lambda}^{-1}, \quad (84)$$

and for $\tau \in ext_0^1(V)$ it yields

$$\underline{\lambda}\tau = \underline{\lambda} \circ \tau \circ \underline{\lambda}^\Delta. \quad (85)$$

Let λ be an invertible linear operator on V^* . Analogously to the above case, it is possible to extend the action of $\underline{\lambda}$ in such a way to get a linear operator on $ext_k^{*l}(V)$. We define

$$ext_k^{*l}(V) \ni \tau \longmapsto \underline{\lambda}\tau \in ext_k^{*l}(V)$$

such that

$$\underline{\lambda}\tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) = \underline{\lambda} \circ \tau(\underline{\lambda}^\Delta(X_1), \dots, \underline{\lambda}^\Delta(X_k), \underline{\lambda}^{-1}(\Phi^1), \dots, \underline{\lambda}^{-1}(\Phi^l)) \quad (86)$$

for each $X_1, \dots, X_k \in \bigwedge V$ and $\Phi^1, \dots, \Phi^l \in \bigwedge V^*$.

For instance, for $\tau \in ext_1^0(V)$ the above definition yields

$$\underline{\lambda}\tau = \underline{\lambda} \circ \tau \circ \underline{\lambda}^\Delta, \quad (87)$$

and for $\tau \in ext_0^1(V)$ it implies that

$$\underline{\lambda}\tau = \underline{\lambda} \circ \tau \circ \underline{\lambda}^{-1}. \quad (88)$$

We give some of the properties of the action of the extended of a vector operator λ on multivector extensors.

- For all $\tau \in \text{ext}_k^l(V)$ and $\sigma \in \text{ext}_r^s(V)$:

$$\underline{\lambda}(\tau \wedge \sigma) = (\underline{\lambda}\tau) \wedge (\underline{\lambda}\sigma). \quad (89)$$

- For all $\tau \in \text{ext}_k^{*l}(V)$ and $\sigma \in \text{ext}_r^s(V)$:

$$\underline{\lambda} \langle \tau, \sigma \rangle = \langle \underline{\lambda}^{-\Delta} \tau, \underline{\lambda} \sigma \rangle, \quad (90)$$

$$\underline{\lambda} \langle \tau, \sigma | = \langle \underline{\lambda}^{-\Delta} \tau, \underline{\lambda} \sigma |, \quad (91)$$

$$\underline{\lambda} | \sigma, \tau \rangle = | \underline{\lambda} \sigma, \underline{\lambda}^{-\Delta} \tau \rangle. \quad (92)$$

We present the proof of the property given by Eq.(91), the other proofs are analogous.

Proof. Without any loss of generality, we give the proof for the particular case where $\tau \in \text{ext}(\bigwedge_1^\diamond V, \bigwedge_2^\diamond V^*; \bigwedge^\diamond V^*)$ and $\sigma \in \text{ext}(\bigwedge_3^\diamond V, \bigwedge_4^\diamond V^*; \bigwedge^\diamond V)$. Take $X \in \bigwedge_1^\diamond V$, $Y \in \bigwedge_3^\diamond V$ and $\Phi \in \bigwedge_2^\diamond V^*$, $\Psi \in \bigwedge_4^\diamond V^*$. A straightforward calculation, using Eq.(83), Eq.(50), Eq.(78) and Eq.(86), gives

$$\begin{aligned} \underline{\lambda} \langle \tau, \sigma | (X, Y, \Phi, \Psi) &= \underline{\lambda} \circ \langle \tau, \sigma | (\underline{\lambda}^{-1}(X), \underline{\lambda}^{-1}(Y), \underline{\lambda}^\Delta(\Phi), \underline{\lambda}^\Delta(\Psi)) \\ &= \underline{\lambda} \langle \tau(\underline{\lambda}^{-1}(X), \underline{\lambda}^\Delta(\Phi)), \sigma(\underline{\lambda}^{-1}(Y), \underline{\lambda}^\Delta(\Psi)) | \\ &= \langle \underline{\lambda}^{-\Delta} \circ \tau(\underline{\lambda}^{-1}(X), \underline{\lambda}^\Delta(\Phi)), \underline{\lambda} \circ \sigma(\underline{\lambda}^{-1}(Y), \underline{\lambda}^\Delta(\Psi)) | \\ &= \langle \underline{\lambda}^{-\Delta} \tau(X, \Phi), \underline{\lambda} \sigma(Y, \Psi) | \\ &= \langle \underline{\lambda}^{-\Delta} \tau, \underline{\lambda} \sigma | (X, Y, \Phi, \Psi), \end{aligned}$$

whence, the expected result follows. ■

We present now some of the properties of the action of the extended of a form operator λ on multiform extensors.

- For all $\tau \in \text{ext}_k^{*l}(V)$ and $\sigma \in \text{ext}_r^{*s}(V)$:

$$\underline{\lambda}(\tau \wedge \sigma) = (\underline{\lambda}\tau) \wedge (\underline{\lambda}\sigma). \quad (93)$$

- For all $\tau \in \text{ext}_k^{*l}(V)$ and $\sigma \in \text{ext}_r^s(V)$:

$$\underline{\lambda} \langle \tau, \sigma \rangle = \langle \underline{\lambda}\tau, \underline{\lambda}^{-\Delta} \sigma \rangle, \quad (94)$$

$$\underline{\lambda} \langle \tau, \sigma | = \langle \underline{\lambda}\tau, \underline{\lambda}^{-\Delta} \sigma |, \quad (95)$$

$$\underline{\lambda} | \sigma, \tau \rangle = | \underline{\lambda}^{-\Delta} \sigma, \underline{\lambda}\tau \rangle. \quad (96)$$

8 Generalization Procedure for Operators

Let γ be a linear operator on V , i.e., a linear map $V \ni v \mapsto \gamma(v) \in V$. It can be generalized in such a way to give a linear operator on $\bigwedge V$, namely γ , which is defined by

$$\bigwedge V \ni X \mapsto \gamma(X) \in \bigwedge V$$

such that

$$\gamma(X) = \gamma(e_j) \wedge \langle \varepsilon^j, X \rangle, \quad (97)$$

where $\{e_j, \varepsilon^j\}$ is any pair of dual bases for V and V^* .

We note that the multivector $\gamma(e_j) \wedge \langle \varepsilon^j, X \rangle$ does not depend on the choice of $\{e_j, \varepsilon^j\}$, and that the linearity of the duality contracted product implies the linearity of γ . Thus, γ is a well-defined linear operator on $\bigwedge V$.

We call γ the *generalized of γ* (to multivector operator).

The generalized of a vector operator γ has the following basic properties.

- γ is grade-preserving, i.e.,

$$\text{if } X \in \bigwedge^k V, \text{ then } \gamma(X) \in \bigwedge^k V. \quad (98)$$

- For all $\alpha \in \mathbb{R}$, $v \in V$, and $X, Y \in \bigwedge V$:

$$\gamma(\alpha) = 0, \quad (99)$$

$$\gamma(v) = \gamma(v), \quad (100)$$

$$\gamma(X \wedge Y) = \gamma(X) \wedge Y + X \wedge \gamma(Y). \quad (101)$$

The four properties given by Eq.(98), Eq.(99), Eq.(100) and Eq.(101) are completely equivalent to the generalization procedure for vector operators.

Let γ be a linear operator on V^* , i.e., a linear map $V^* \ni \omega \mapsto \gamma(\omega) \in V^*$. It is possible to generalize γ in such a way as to get a linear operator on $\bigwedge V^*$, namely γ , which is defined by

$$\bigwedge V^* \ni \Phi \mapsto \gamma(\Phi) \in \bigwedge V^*$$

such that

$$\gamma(\Phi) = \gamma(\varepsilon^j) \wedge \langle e_j, \Phi |, \quad (102)$$

where $\{e_j, \varepsilon^j\}$ is any pair of dual bases over V .

We emphasize that γ is a well-defined linear operator on $\bigwedge V^*$, and call it the *generalized of γ* (to a multiform operator).

The generalized of a form operator γ has the following basic properties.

- γ is grade-preserving, i.e.,

$$\text{if } \Phi \in \bigwedge^k V^*, \text{ then } \gamma(\Phi) \in \bigwedge^k V^*. \quad (103)$$

- For all $\alpha \in \mathbb{R}$, $\omega \in V^*$, and $\Phi, \Psi \in \bigwedge V^*$ we have

$$\gamma(\alpha) = 0, \quad (104)$$

$$\gamma(\omega) = \gamma(\omega), \quad (105)$$

$$\gamma(\Phi \wedge \Psi) = \gamma(\Phi) \wedge \Psi + \Phi \wedge \gamma(\Psi). \quad (106)$$

The properties given by Eq.(103), Eq.(104), Eq.(105) and Eq.(106) are logically equivalent to the generalization procedure for *form* operators.

There exists a relationship between the generalization procedure of a vector operator and the generalization procedure of a form operator.

Let γ a vector operator (or, a form operator). As we already know, the duality adjoint of γ is just a form operator (respectively, a vector operator), and the duality adjoint of γ is just a multiform operator (respectively, a multivector operator). The duality adjoint of the generalized of γ is equal to the generalized of the duality adjoint of γ , i.e.,

$$\left(\gamma \right)^\Delta = (\gamma^\Delta). \quad (107)$$

It follows that is possible to use a more simple notation, namely γ^Δ to mean either $\left(\gamma \right)^\Delta$ or (γ^Δ) .

We give some of the main properties of the generalized of a vector operator γ .

- For all $\Phi \in \wedge V^*$, and $X \in \wedge V$:

$$\gamma \langle \Phi, X \rangle = - \left\langle \gamma^\Delta(\Phi), X \right\rangle + \left\langle \Phi, \gamma(X) \right\rangle, \quad (108)$$

$$\gamma \langle \Phi, X | = - \left\langle \gamma^\Delta(\Phi), X \right| + \left\langle \Phi, \gamma(X) \right|, \quad (109)$$

$$\gamma |X, \Phi \rangle = \left| \gamma(X), \Phi \right\rangle - \left| X, \gamma^\Delta(\Phi) \right\rangle. \quad (110)$$

We prove the property given by Eq.(109), the other proofs are analogous.

Proof. Let us take $X \in \wedge V$ and $\Phi, \Psi \in \wedge V^*$. A straightforward calculation, by using Eq.(57), Eq.(27), Eq.(103) and Eq.(106), yields

$$\begin{aligned} \left\langle \gamma \langle \Phi, X |, \Psi \right\rangle &= \left\langle \langle \Phi, X |, \gamma^\Delta(\Psi) \right\rangle = \left\langle X, \tilde{\Phi} \wedge \gamma^\Delta(\Psi) \right\rangle \\ &= \left\langle X, -\widetilde{\gamma^\Delta(\Phi)} \wedge \Psi + \gamma^\Delta(\tilde{\Phi}) \wedge \Psi + \tilde{\Phi} \wedge \gamma^\Delta(\Psi) \right\rangle \\ &= - \left\langle \left\langle \gamma^\Delta(\Phi), X \right|, \Psi \right\rangle + \left\langle X, \gamma^\Delta(\tilde{\Phi} \wedge \Psi) \right\rangle \\ &= - \left\langle \left\langle \gamma^\Delta(\Phi), X \right|, \Psi \right\rangle + \left\langle \left\langle \Phi, \gamma(X) \right|, \Psi \right\rangle \\ &= \left\langle - \left\langle \gamma^\Delta(\Phi), X \right| + \left\langle \Phi, \gamma(X) \right|, \Psi \right\rangle, \end{aligned}$$

and by the non-degeneracy of duality scalar product, the expected result follows. ■

We present some properties for the generalized of a form operator γ .

- For all $\Phi \in \wedge V^*$, and $X \in \wedge V$:

$$\gamma \langle \Phi, X \rangle = \left\langle \gamma(\Phi), X \right\rangle - \left\langle \Phi, \gamma^\Delta(X) \right\rangle, \quad (111)$$

$$\gamma \langle \Phi, X | = \left\langle \gamma(\Phi), X \right| - \left\langle \Phi, \gamma^\Delta(X) \right|, \quad (112)$$

$$\gamma |X, \Phi \rangle = - \left| \gamma^\Delta(X), \Phi \right\rangle + \left| X, \gamma(\Phi) \right\rangle. \quad (113)$$

8.1 Action of Generalized Operators on Extensors

Let γ be a linear operator on V . We can generalize the action of γ in such a way γ is to map multivector extensors over V into multivector extensors over V . We define the linear mapping

$$ext_k^l(V) \ni \tau \longmapsto \gamma\tau \in ext_k^l(V)$$

such that

$$\begin{aligned} \gamma\tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) &= \gamma \circ \tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) \\ &\quad - \tau(\gamma(X_1), \dots, X_k, \Phi^1, \dots, \Phi^l) \\ &\quad \cdots - \tau(X_1, \dots, \gamma(X_k), \Phi^1, \dots, \Phi^l) \\ &\quad + \tau(X_1, \dots, X_k, \gamma^\Delta(\Phi^1), \dots, \Phi^l) \\ &\quad \cdots + \tau(X_1, \dots, X_k, \Phi^1, \dots, \gamma^\Delta(\Phi^l)) \end{aligned} \quad (114)$$

for each $X_1, \dots, X_k \in \bigwedge V$ and $\Phi^1, \dots, \Phi^l \in \bigwedge V^*$.

Thus, we can think of γ as a linear multivector extensor operator.

For instance, for $\tau \in ext_1^0(V)$ this definition above gives

$$\gamma\tau = \gamma \circ \tau - \tau \circ \gamma = \left[\gamma, \tau \right], \quad (115)$$

and for $\tau \in ext_0^1(V)$ it yields

$$\gamma\tau = \gamma \circ \tau + \tau \circ \gamma^\Delta. \quad (116)$$

Let γ be an invertible linear operator on V^* . Analogously to the case above, it is possible to generalize the action of γ in such a way to get a

linear operator on $ext_k^*{}^l(V)$. We define

$$ext_k^*{}^l(V) \ni \tau \longmapsto \gamma\tau \in ext_k^*{}^l(V)$$

such that

$$\begin{aligned}
\gamma \tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) &= \gamma \circ \tau(X_1, \dots, X_k, \Phi^1, \dots, \Phi^l) \\
&\quad + \tau(\gamma^\Delta(X_1), \dots, X_k, \Phi^1, \dots, \Phi^l) \\
&\quad \cdots + \tau(X_1, \dots, \gamma^\Delta(X_k), \Phi^1, \dots, \Phi^l) \\
&\quad - \tau(X_1, \dots, X_k, \gamma(\Phi^1), \dots, \Phi^l) \\
&\quad \cdots - \tau(X_1, \dots, X_k, \Phi^1, \dots, \gamma(\Phi^l)) \quad (117)
\end{aligned}$$

for each $X_1, \dots, X_k \in \bigwedge V$ and $\Phi^1, \dots, \Phi^l \in \bigwedge V^*$.

For instance, for $\tau \in \text{ext}_1^{*0}(V)$ the definition just given above yields

$$\gamma \tau = \gamma \circ \tau + \tau \circ \gamma^\Delta, \quad (118)$$

and for $\tau \in \text{ext}_0^{*1}(V)$ it holds

$$\gamma \tau = \gamma \circ \tau - \tau \circ \gamma = [\gamma, \tau]. \quad (119)$$

We present some of the main properties of the action of the generalized operator of a vector operator γ on multivector extensors.

- For all $\tau \in \text{ext}_k^l(V)$ and $\sigma \in \text{ext}_r^s(V)$:

$$\gamma(\tau \wedge \sigma) = (\gamma \tau) \wedge \sigma + \tau \wedge (\gamma \sigma). \quad (120)$$

- For all $\tau \in \text{ext}_k^{*l}(V)$ and $\sigma \in \text{ext}_r^s(V)$:

$$\gamma \langle \tau, \sigma \rangle = - \left\langle \gamma^\Delta \tau, \sigma \right\rangle + \left\langle \tau, \gamma \sigma \right\rangle, \quad (121)$$

$$\gamma \langle \tau, \sigma | = - \left\langle \gamma^\Delta \tau, \sigma \right| + \left\langle \tau, \gamma \sigma \right|, \quad (122)$$

$$\gamma | \sigma, \tau \rangle = \left| \gamma \sigma, \tau \right\rangle - \left| \sigma, \gamma^\Delta \tau \right\rangle. \quad (123)$$

Proof. We present only the proof for the property given by Eq. (120), the others are similar. Without any loss of generality, we give the proof for the particular case where $\tau \in \text{ext}(\bigwedge_1^\diamond V, \bigwedge_2^\diamond V^*; \bigwedge^\diamond V^*)$ and $\sigma \in \text{ext}(\bigwedge_3^\diamond V, \bigwedge_4^\diamond V^*; \bigwedge^\diamond V)$. Take $X \in \bigwedge_1^\diamond V$, $Y \in \bigwedge_3^\diamond V$ and $\Phi \in \bigwedge_2^\diamond V^*$, $\Psi \in \bigwedge_4^\diamond V^*$. By using the definition of γ , we have

$$\begin{aligned} \gamma(\tau \wedge \sigma)(X, Y, \Phi, \Psi) &= \gamma \circ (\tau \wedge \sigma)(X, Y, \Phi, \Psi) - (\tau \wedge \sigma) \left(\gamma X, Y, \Phi, \Psi \right) \\ &\quad - (\tau \wedge \sigma) \left(X, \gamma Y, \Phi, \Psi \right) + (\tau \wedge \sigma) \left(X, Y, \gamma \Phi, \Psi \right) \\ &\quad + (\tau \wedge \sigma) \left(X, Y, \Phi, \gamma \Psi \right). \end{aligned} \tag{124}$$

Now, using the property (101) we can write the first term of right side of the Eq. (124) as

$$\gamma \circ (\tau \wedge \sigma)(X, Y, \Phi, \Psi) = \gamma \circ \tau(X, \Phi) \wedge \sigma(Y, \Psi) + \tau(X, \Phi) \wedge \gamma \circ \sigma(Y, \Psi),$$

and remembering that

$$(\tau \wedge \sigma)(X, Y, \Phi, \Psi) = \tau(X, \Phi) \wedge \sigma(Y, \Psi),$$

the Eq. (124) can be written as

$$\begin{aligned} \gamma(\tau \wedge \sigma)(X, Y, \Phi, \Psi) &= \left[\gamma \circ \tau(X, \Phi) - \tau(\gamma X, \Phi) + \tau(X, \gamma^\Delta \Phi) \right] \wedge \sigma(Y, \Psi) \\ &\quad + \tau(X, \Phi) \wedge \left[\gamma \circ \sigma(Y, \Psi) - \sigma(\gamma Y, \Psi) + \sigma(Y, \gamma^\Delta \Psi) \right] \end{aligned}$$

or

$$\begin{aligned} \gamma(\tau \wedge \sigma)(X, Y, \Phi, \Psi) &= \left[\left(\gamma \tau \right)(X, \Phi) \right] \wedge \sigma(Y, \Psi) + \tau(X, \Phi) \wedge \left[\left(\gamma \sigma \right)(Y, \Psi) \right] \\ &= \left(\gamma \tau \wedge \sigma + \tau \wedge \gamma \sigma \right)(X, Y, \Phi, \Psi), \end{aligned}$$

and the property is proved. ■

We present some properties for the action of the generalized operator of a form operator γ on multiform extensors.

- For all $\tau \in \text{ext}_k^{*l}(V)$ and $\sigma \in \text{ext}_r^{*s}(V)$:

$$\gamma(\tau \wedge \sigma) = (\gamma\tau) \wedge \sigma + \tau \wedge (\gamma\sigma). \quad (125)$$

- For all $\tau \in \text{ext}_k^{*l}(V)$ and $\sigma \in \text{ext}_r^{*s}(V)$:

$$\gamma \langle \tau, \sigma \rangle = \left\langle \gamma\tau, \sigma \right\rangle - \left\langle \tau, \gamma^\Delta \sigma \right\rangle, \quad (126)$$

$$\gamma \langle \tau, \sigma | = \left\langle \gamma\tau, \sigma \right| - \left\langle \tau, \gamma^\Delta \sigma \right|, \quad (127)$$

$$\gamma |\sigma, \tau \rangle = - \left| \gamma^\Delta \sigma, \tau \right\rangle + \left| \sigma, \gamma\tau \right\rangle. \quad (128)$$

9 Conclusions

In this paper we studied the properties of the *duality product* of multivectors and multiforms (used for the definition of the hyperbolic Clifford algebra of *multivefors* [5]) introducing a very useful notation for the left and right contracted products of multiforms (elements of $\bigwedge V^*$) and multivectors (elements of $\bigwedge V$). Next, we introduced a theory of the *k multivector and l multiform variables multivector (or multiform) extensors* over V (defining the spaces $\text{ext}_k^{*l}(V)$ and $\text{ext}_k^{*l}(V)$) defining the exterior product of extensors, and several important operations on them. This algebraic theory will play an important role in a presentation of the differential geometry of a manifold M of arbitrary topology discussed in forthcoming papers.

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